Calculation of Supersonic Flows with Strong Viscous-Inviscid Interaction

Mark Barnett* and R. Thomas Davis†
University of Cincinnati, Cincinnati, Ohio

A procedure for the calculation of strong viscous-inviscid interactions in two-dimensional laminar supersonic flows with and without separation is described. The equations solved are the so-called parabolized Navier-Stokes equations. The streamwise pressure gradient term is written as a combination of a forward and a backward difference to provide a path for upstream propagation of information. Global iteration is utilized to repeatedly update the pressure field from an initial guess until convergence is achieved. The numerical scheme employed is a new alternating direction explicit (ADE) procedure, which is used as an alternative to the more difficult to program multigrid strategy to accelerate convergence. Results are presented for flows past two flat plate related bodies.

Introduction

THE calculation of supersonic flows in which rapid changes in geometry or boundary condition occur often requires techniques other than classical Prandtl boundary-layer theory or other solution methods suitable for weakly interacting flows. The reason for this is that at high Reynolds number, rapid changes in geometry or boundary condition induce large changes in the properties of the boundary layer over a very short length scale. When this occurs, the phenomenon of upstream influence, wherein information is propagated in the upstream direction, becomes important. This type of flow is referred to as a "strongly interacting" flow. Methods that treat such flows as initial-value problems are not suitable because they are boundary value in nature, in the predominant flow direction.

In this paper, a procedure for two-dimensional supersonic strongly interacting flows is developed, which utilizes the parabolized or reduced Navier-Stokes equations. We will employ the former term here. For simplicity it will be assumed that x and y are nearly streamwise and stream normal directions, respectively. With special treatment of the pressure gradient term $\partial p/\partial x$, in the streamwise momentum equation, one can develop a procedure which is well suited for calculating weakly interacting flows.[1-4] In order to calculate strongly interacting flows, a means for propagating information in the upstream direction is required. This can be satisfied by utilizing a forward difference for the streamwise pressure gradient. In subsonic flows, where upstream influence is important even if the viscous-inviscid interaction is weak, a forward difference of p_x has been shown by Rubin[5] to allow the calculation of such flows. Lin and Rubin[6] later successfully applied forward differencing of p_x to a supersonic flow with weak interaction. In that study, forward differencing was utilized throughout the flowfield, including the supersonic flow. In the present technique forward differencing of p_x is applied only in the subsonic portion of the flowfield, which leads to more rapid convergence than the use of a full forward difference everywhere.

Results are presented for flows over a compression ramp and a parabolic bump on a flat plate. Comparisons with results obtained using other techniques are provided.

Governing Equations and Boundary Conditions

Two techniques commonly utilized to calculate flows with strong viscous-inviscid interaction are 1) to solve the full Navier-Stokes equations or, 2) to employ the interacting boundary-layer equations.

The solution of the full Navier-Stokes equations generally entails a large amount of computer time and storage. The equations are most often solved in unsteady form, marching them in time until a steady state solution is achieved, if one exists. This usually requires a large number of time steps. Storage requirements are substantial because all unknowns must be saved at every grid point, usually at two or more time levels. In many problems, terms are retained which are negligible from an order of magnitude standpoint. This is inefficient, mainly because the neglect of such terms can lead to a system of equations solvable by more efficient techniques.

The interacting boundary-layer equations suffer from neither of the above disadvantages. However, if one wishes to solve for supersonic flows wherein complex inviscid phenomena are occurring, such as imbedded shocks, the interacting boundary-layer equations in conjunction with a pressure/displacement thickness interaction law are not suitable. In addition, the most commonly solved forms of the interacting boundary-layer equations do not include normal pressure gradients. This approximation is appropriate for most flows unless one is considering a geometry with a small radius of curvature or turbulent flows, cases for which normal pressure gradients may become significant. This poses no particular difficulty, however, since inclusion of the normal gradient in interacting boundary-layer theory is possible. For the aforementioned reasons, an alternative method for strong interaction problems based upon the parabolized Navier-Stokes (PNS) equations is developed in this study.

The parabolized Navier-Stokes equations are obtained directly from the full Navier-Stokes equations by deleting appropriate terms. The deleted terms are determined from order of magnitude considerations and from consideration of the mathematical nature of the equations. The ultimate

Presented as Paper 85-0016 at the AIAA 23rd Aerospace Sciences Meeting, Reno, NV, Jan. 14-17, 1985; received Feb. 7, 1985; revision received March 17, 1986.

*Research Assistant, Department of Aerospace Engineering and Applied Mechanics; presently Associate Research Engineer, United Technologies Research Center, East Hartford, CT. Member AIAA.

†Professor, Department of Aerospace Engineering and Applied Mechanics (deceased). Associate Fellow AIAA.

aim is to develop a composite set of equations which is valid in both the viscous and inviscid portions of the flowfield.

The PNS equations are obtained from the full Navier-Stokes equations by assuming the flow to be steady and neglecting streamwise derivatives of viscous terms. The physical interpretation of the latter approximation is that streamwise diffusion is neglected in comparison to normal diffusion. To comparable order of magnitude, it is consistent to neglect all viscous terms in the normal momentum equation. The assumptions made with regard to the viscous terms correspond to those which lead to the boundary-layer equations.

The modifications to the previously described Navier-Stokes equations constitute the usual parabolizing assumptions. The final approximation made to the governing equations is required when solving for weakly interacting flows to remove the ellipticity, which is still inherent in the parabolized Navier-Stokes equations. This ellipticity is due to the streamwise pressure gradient term and its role in a strong viscous-inviscid interaction. See Stewartson,[7] Davis and Werle,[8] and Barnett,[9] for further discussion of this point.

The final approximation consists of splitting $\partial p/\partial x$ into two terms

$$\frac{\partial p}{\partial x} = \omega\frac{\partial p}{\partial x} + (1-\omega)\frac{\partial p}{\partial x} \tag{1}$$

and dropping the last term. The factor ω is determined in such a way that the weak interaction procedure is well posed, that is, the equations are mathematically non-elliptic. The appropriate form for ω was determined by Vigneron et al.[4] to be

$$\omega = \frac{\gamma M^2}{1+(\gamma-1)M^2} \qquad \text{for} \quad M<1 \tag{2a}$$

$$\omega = 1 \qquad \text{for} \quad M \geq 1 \tag{2b}$$

where M is the Mach number component in the x direction. When solving for flows where the viscous-inviscid interaction is strong, the term $(1-\omega)\partial p/\partial x$ is retained and forward differenced, thus providing a path for upstream propagation of information.

The governing equations, after application of the parabolizing assumptions, splitting of the streamwise pressure gradient term, and transformation to an appropriate coordinate system, can be found in Ref. 9. For simplicity, we will assume that we can work in Cartesian coordinates where x and y are nearly streamwise and stream normal coordinates, respectively. In this form the equations can be written as

$$\frac{\partial E^*}{\partial x} + \frac{\partial P}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F_v}{\partial y} = 0 \tag{3}$$

where

$$E^* = [\rho u, \rho u^2 + \omega p, \rho uv, (\rho e_t + p)u]^T \tag{4a}$$

$$P = [0, (1-\omega)p, 0, 0]^T \tag{4b}$$

$$F = [\rho v, \rho uv, \rho v^2 + p, (\rho e_t + p)v]^T \tag{4c}$$

$$F_v = [0, \tau_{xy}, 0, u\tau_{xy} + v\sigma_{yy} + q_y]^T \tag{4d}$$

representing the continuity, x momentum, y momentum, and energy equations, respectively, in the vector notation employed here. The quantities u and v are the velocities in the x and y directions, respectively, ρ is the density, p is the pressure, σ_{yy} and τ_{xy} are the viscous stress terms given by

$$\sigma_{yy} = 2\mu(2v_y - u_x)/3Re \tag{5a}$$

$$\tau_{xy} = \mu(u_y + v_x)/Re \tag{5b}$$

(prior to application of the parabolizing assumptions) and q_y is the heat flux term,

$$q_y = \frac{\mu}{Re}\frac{1}{Pr(\gamma-1)M_\infty^2}T_y \tag{6}$$

where T is the temperature and Pr is the Prandtl number. All quantities are nondimensionalized using the standard nondimensionalizations for high speed flow.[9] The Reynolds number, Re, is defined by

$$Re = \rho_\infty^* U_\infty^* L^*/\mu_\infty^* \tag{7}$$

where (*) denotes dimensional quantities. Closure is obtained by employing the equation of state and using the Sutherland viscosity law to relate the molecular viscosity to the temperature.

The boundary conditions for the flows considered in this study are given as follows. At the solid surface

$$u(x,y_b) = v(x,y_b) = 0 \tag{8}$$

$$T(x,y_b) = T_w(x) \tag{9a}$$

or

$$\frac{\partial T}{\partial n}(x,y_b) = 0 \tag{9b}$$

In these expressions y_b represents the value of y at the body surface, T_w the specified wall temperature, and $\partial/\partial n$ the derivative in the body normal direction. Two possible wall temperature conditions are thus considered, a specified wall temperature [Eq. (9a)] or an adiabatic wall [Eq. (9b)].

At the outer (inviscid) boundary, a boundary is developed that provides an alternative to the use of shock fitting[10] or freestream boundary conditions. The outer boundary is assumed to be placed at a location sufficiently far from the body to be in a region where linear theory (small disturbance theory) applies. Linear theory allows one to write

$$u = U_\infty + u' \tag{10}$$

where $u' \ll U_\infty$. Letting $\mathrm{d}f/\mathrm{d}x$ represent the local streamline slope at a boundary point and assuming that $\mathrm{d}f/\mathrm{d}x \ll 1$ it follows that

$$v = U_\infty \frac{\mathrm{d}f}{\mathrm{d}x} \tag{11}$$

Using the small disturbance theory expression for the coefficient of pressure

$$C_p = -2\frac{u'}{U_\infty} \tag{12}$$

and the isentropic relations, the following expressions for the conserved variables are obtained:

$$\rho = 1 - M_\infty^2 u' \tag{13a}$$

$$\rho u = 1 + \lambda^2 u' \tag{13b}$$

$$\rho v = -\lambda u' \tag{13c}$$

$$\rho e_t = \frac{1}{\gamma(\gamma-1)M_\infty^2} + \frac{1}{2} + \left(\frac{\gamma - 2}{\gamma-1} - \frac{M_\infty^2}{2}\right)u' \tag{13d}$$

where $\gamma - (M_\infty^2 - 1)^{1/2}$. All of the variables have been written in terms of one unknown, u', thus constituting three boundary conditions. With this linear theory boundary condition the outer boundary can be placed very near to the edge of the boundary layer for the geometries considered herein since the bodies may all be viewed as perturbed flat plates. One point which should be kept in mind is that one must avoid strong shocks crossing the outer boundary so that the assumptions of linear theory are not violated.

If the present sixth-order system of governing equations is solved using a general lower-upper decomposition procedure developed to solve four second-order equations, then two connection conditions (numerical boundary conditions) must be applied, as discussed in detail in the paper of Barnett, Davis, and Rakich.[10]

Numerical Method

The governing equations, in practice, are transformed to a computational (ξ,η) plane where the body surface maps to a constant coordinate, and finite differencing can be performed over constant mesh intervals. This topic will not be discussed here; one can refer to many papers where this procedure is detailed.[4,9] Once the equations are transformed, the parabolizing assumptions can be introduced in the appropriate streamwise/stream normal coordinates. The equations are then written in finite-difference form. In this study they are solved using an implicit finite-difference scheme with quasilinearization performed about the previous solution station. The numerical method has a truncation error which is $0(\Delta\xi,\Delta\eta^2)$. For details of the finite-difference scheme see Refs. 9 and 10.

It was mentioned earlier that in solving for weakly interacting supersonic flows, the term P in Eq. (3) is dropped to obtain a well-posed initial-value problem. In solving strongly interacting flow problems, this term will be retained and forward differenced, providing a path for upstream propagation of information. In doing so, an iterative method will be required since the downstream pressure field is not known at the outset. A guess will be needed for the subsonic pressure field, which is subsequently updated until a converged solution is obtained. In addition, an outflow boundary condition is necessary to satisfy the boundary-value nature of the strong interaction problem.

We will now develop the alternating direction explicit (ADE) method for use in solving the supersonic strong interaction problem. We note that at the same time as the ADE method was developed for the PNS equations it was applied to the solution of the interacting boundary-layer equations by Davis.[11]

We write the streamwise pressure gradient in the form of Eq. (1), but we append a fictitious unsteady term to permit information to propagate upstream through the subsonic portion of the boundary in a hyperbolic manner.

$$\frac{\partial p}{\partial x}=\omega\frac{\partial p}{\partial x}+(1-\omega)\left[\frac{\partial p}{\partial x}-\frac{\partial p}{\partial t}\right] \tag{14}$$

This procedure is similar to what is done in interacting boundary-layer theory where a fictitious unsteady term is appended to the second derivative of the displacement thickness.[12]

Next, Eq (14) is written as a two step ADE procedure as follows.

First step:

$$\frac{\partial p}{\partial x}=\omega\frac{p_i^*-p_{i-1}^*}{\Delta x}+(1-\omega)\left[\frac{p_{i+1}^n-p_i^n}{\Delta x}-\frac{p_i^*-p_i^n}{\Delta t}\right] \tag{15}$$

In this equation the current solution station is denoted by the index i. The superscript n denotes the old time level and (*) the intermediate time level at which the first step solution is obtained.

Second step:

$$\frac{\partial p}{\partial x}=\omega\frac{p_i^*-p_{i-1}^*}{\Delta x}+(1-\omega)\left[\frac{p_{i+1}^{n+1}-p_i^{n+1}}{\Delta x}-\frac{p_i^{n+1}-p_i^*}{\Delta t}\right] \tag{16}$$

In both the first and second steps of the ADE procedure, the rest of the terms in the PNS equations are written at the (*) level. This allows one to develop the following expression involving only the pressure by equating the relationships given by Eqs. (15) and (16):

$$\frac{p_{i+1}^n-p_i^n}{\Delta x}-\frac{p_i^*-p_i^n}{\Delta t}=\frac{p_{i+1}^{n+1}-p_i^{n+1}}{\Delta x}-\frac{p_i^{n+1}-p_i^*}{\Delta t} \tag{17}$$

Equation (17) is solved as the second step of the ADE procedure by marching from the downstream boundary to the upstream boundary. This simple technique serves to propogate information upstream in a relatively rapid manner. The first ADE step is implemented in the usual way in which the weak interaction (one pass) PNS procedure is, sweeping from the inflow station downstream to the outflow boundary, with the streamwise pressure gradient term given here by Eq. (15).

To completely specify the strong interaction problem, an outflow boundary condition is required for the pressure. Two alternatives are considered here. Both assume that the outflow boundary is sufficiently far downstream to be in a weakly interacting region of the flow.

The first outflow boundary condition assumes that the streamwise pressure gradient is zero at the outflow boundary. This is implemented by setting the forward difference of the pressure gradient equal to zero. If the zero pressure gradient condition is combined with Eq. (17) at the outflow boundary, we obtain a zero pressure gradient with the pressure level effectively extrapolated in time.

The second outflow boundary condition assumes that the displacement thickness δ^* grows such that

$$\delta^*\propto\sqrt{x} \tag{18}$$

which is the result of classical Prandtl boundary-layer theory. Using Eq. (18) and the relationships of linear theory after defining the nondimensional pressure as $p=p_\infty^*\div\rho_\infty^*U_\infty^{*2}$, we may write

$$p=-2xp_x+\frac{1}{\gamma M_\infty^2}+\frac{1}{\lambda}\left(\frac{dy_b}{dx}+2x\frac{d^2y_b}{dx^2}\right) \tag{19}$$

where $\lambda=(M_\infty^2-1)^{1/2}$ and dy_b/dx and d^2y_b/dx^2 are the body surface slope and second derivative, respectively, at the outflow boundary. Equation (19) can be combined with Eq. (17) evaluated at the outflow boundary to solve for p there, giving the final form of this outflow boundary condition.

It was found that either of the two outflow boundary conditions can be used with virtually no difference between the resulting solutions in the strong interaction region of the flow, provided the boundary is far enough downstream.

A few notes are appropriate at this point regarding the application of the ADE method. First, because $\partial p/\partial t$ is multiplied by $(1-\omega)$, the unsteady term does not contribute in the first step of the ADE procedure wherever the flow is supersonic. Second, ω can be evaluated either from the previous iteration at i or lagged at $i-1$ using the current iteration value. This should not be critical, provided ω is not varying rapidly with x. Third, although Eq. (3) is not written to reflect this, we do not split p itself using ω. In the numerical scheme we split p_x as expressed through Eq. (1). In addition, just as the outflow boundary is placed downstream in a weakly interacting region of the flow, the upstream boundary should be placed in an upstream region of weak interaction.

It is important to recognize that, since we are solving for strongly interacting laminar flows, the scalings which apply are those which govern the triple deck.[7,13,14] Thus, the mesh employed for the numerical calculations should be specified so as to honor those scalings. Specifically, the streamwise scaling is of $0(Re^{-3/8})$ and the normal scaling at the wall is of $0(Re^{-5/8})$.

It is desirable to be able to calculate separated flows with the present computational method since they constitute an important class of strongly interacting flows. Solution methods for the PNS equations are known to be unstable when forward marching in regions where u is negative. There are several alternatives for stabilizing the solution. The most accurate procedure is to forward difference convective terms when u is negative. For the flat plate related bodies considered here, the reversed flow velocities are always small. Therefore, simpler approximations were employed for convective terms in this study. Three alternatives were considered. When u is negative we can either set $u=|u|$, $u=0$, or drop all convective terms. The last alternative may appear to be a severe approximation, however Davis[15] showed it to be reasonable provided the negative u velocities remain small throughout separated regions. All three methods were found to give similar results for the cases considered herein.

One final point is considered before discussing the numerical results obtained in this study. It is necessary to modify the forward difference of the streamwise pressure gradient term when there is a discontinuity in the body slope, as in the sharp compression ramp and bump geometries.

For the actual calculations, the governing equations are transformed to strong conservation form. The mesh used is a skewed Cartesian mesh, illustrated for a compression ramp in Fig. 1. The ξ coordinate lines have a slope discontinuity at $x=1.0$. The ξ coordinate is assumed to be nearly streamwise so that in practice, p_ξ is split as in Eq. (1) instead of p_x. Since a fraction of p_ξ is finite differenced across the coordinate discontinuity, a "jump condition" is required.

The appropriate jump condition is obtained in the following manner, assuming that p, p_x, and p_y are continuous across the discontinuity. In strong conservation form we may write

$$p_x = J\left[\left(\xi_x \frac{p}{J}\right)_\xi + \left(\eta_x \frac{p}{J}\right)_\eta\right] \tag{20a}$$

$$p_y = J\left[\left(\xi_y \frac{p}{J}\right)_\xi + \left(\eta_y \frac{p}{J}\right)_\eta\right] \tag{20b}$$

where J is the Jacobian of the geometrical transformation and is continuous across the coordinate discontinuity. Letting $(-)$ and $(+)$ denote the locations immediately upstream and downstream of the corner, respectively, we may write

$$p_x^+ = p_x^- \tag{21a}$$

$$p_y^+ = p_y^- \tag{21b}$$

For the present coordinate system $\xi = \xi(x)$, therefore, one can obtain the relationship

$$\left(\xi_x \frac{p}{J}\right)_\xi^- = \left(\xi_x \frac{p}{J}\right)_\xi^+ - (y_{b,\xi}^+ - y_{b,\xi}^-)p_\eta \tag{22}$$

where the last term constitutes the jump condition across the corner and is appended to the bracketed terms in Eq. (14) at the corner station. The quantity $y_{b,\xi}^+ - y_{b,\xi}^-$ represents the jump in the body slope across the corner. See Ref. 9 for a more detailed discussion of the jump condition.

One can apply the jump condition at the (*) level in both steps of the ADE procedure. In doing so, the jump condition will appear only in the first ADE step (implicitly), dropping out of the second when Eqs. (15) and (16) are combined to obtain Eq. (17).

The application of the ADE procedure can be summarized in the following steps:

1) Guess subsonic pressure field.

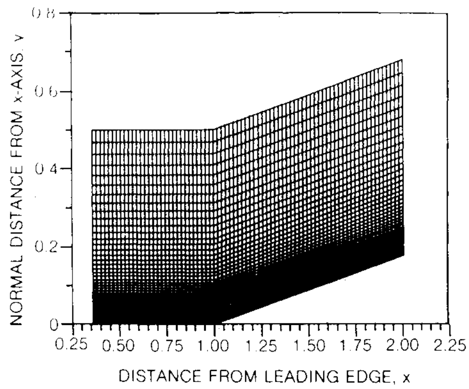

Fig. 1 Typical grid for compression corner problems.

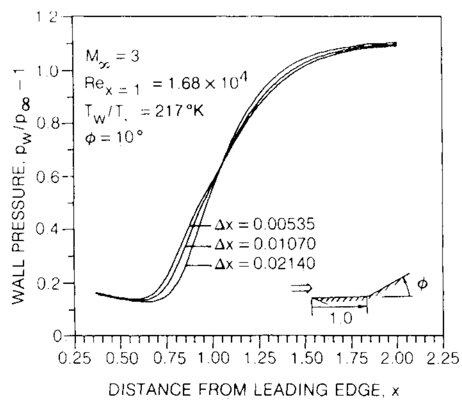

Fig. 2 Sharp compression ramp results; effects of stepsize, p_w vs x.

2) Solve PNS equations with streamwise pressure gradient given by Eq. (15). The solution is marched downstream from the upstream boundary. Pressure at the intermediate time level p^* is obtained.

3) Solve for pressure at the latest time level, p^{n+1}. Solution is obtained by solving Eq. (17) starting at the downstream boundary and marching upstream.

4) Check the solution for convergence by monitoring the magnitude of the difference between p^n and p^{n+1} in the subsonic flowfield. If it is found to be above a specified tolerance, steps 2–4 are repeated.

Results

The present ADE solution method for the parabolized Navier-Stokes equations was employed to calculate the laminar supersonic flows over two geometries. The geometries considered are a compression ramp with a sharp corner and a parabolic bump on a flat plate.

The compression ramp results for wall pressure and skin friction are shown in Figs. 2 and 3 for three stepsizes, Δx. The conditions for this case are a freestream Mach number of 3.0, a Reynolds number based on the corner location of 1.68×10^4, $T_w/T_\infty = 2.8$, and a freestream temperature of 390°R. For this and all subsequent cases the molecular Prandtl number is assumed to be equal to 0.72 and the ramp angle is 10 deg. The sensitivity of the solution to Δx is evident in these figures. However, as $\Delta x \to 0$, the results show the typical behavior of separated compression corner flows with a tendency toward a pressure plateau at the corner and a secondary decrease in wall shear after the corner. Figures 4 and 5 show a comparison of the PNS solution on the finest mesh ($\Delta x = 0.00535$) with the second-order accurate full

Navier-Stokes solution of Carter.[16] The qualitative agreement is good. Extrapolation of the first-order accurate PNS solutions to zero stepsize (second-order accuracy) would probably yield good quantitative agreement, as well.

The full Navier-Stokes solution of Carter[16] was obtained using a constant Δx of 0.0214 and 28 grid points in the body normal direction. Convergence to three decimal place accuracy was obtained in approximately 2200 time steps. The PNS solution obtained on the finest mesh employed 55 normal grid points and converged to a maximum relative change in the pressure of 1×10^{-6} in an order of magnitude fewer iterations than the full Navier-Stokes solution, approximately 300 iterations.

Supersonic flow over a parabolic bump was also calculated. The flow conditions are $M_\infty = 2.0$, $Re = 5$ million based on the starting location of the bump, an insulated wall, and a freestream temperature of 300°R. The bump geometry is given by

$$y_b(x) = 0 \qquad \text{for } x \leq 1$$
$$y_b(x) = h(x-1)(c-x) \qquad \text{for } 1 < x \leq c$$
$$y_b(x) = 0 \qquad \text{for } x > c \qquad (23)$$

where $c = 1.02$ for the present case. The value of c is such that a triple deck disturbance is provoked for the flow conditions and bump heights h, chosen.

Figures 6 and 7 show the wall pressure and wall shear distributions for various values of h between 0 and 5. As h is increased the flow separates at the trailing edge. If h were further increased, leading-edge separation would also occur. Figures 8 and 9 show the comparison with a first-order accurate interacting boundary-layer result of Davis[11] for $h = 1$. Qualitative agreement is observed. When solutions using both methods were obtained on a series of finer meshes, better quantitative agreement was found as both solutions appear to be approaching approximately the same limiting solution. All of the bump results were obtained with $\Delta x = 0.001$.

Figure 10 shows the ADE convergence histories for a series of compression ramp calculations at various ramp angles.

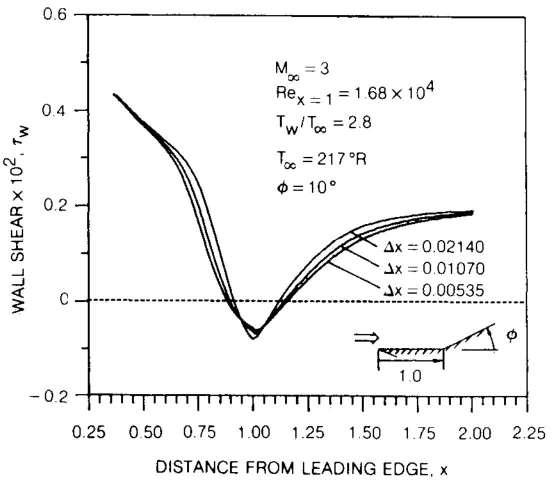

Fig. 3 Sharp compression ramp results; effects of stepsize, τ_w vs x.

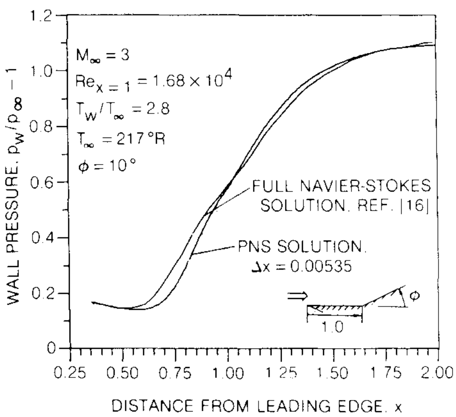

Fig. 4 Sharp compression ramp; comparison of PNS and full Navier-Stokes solutions, p_w vs x.

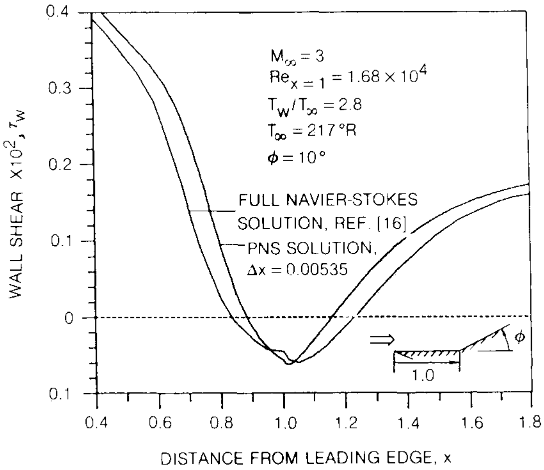

Fig. 5 Sharp compression ramp; comparison of PNS and Navier-Stokes solutions, τ_w vs x.

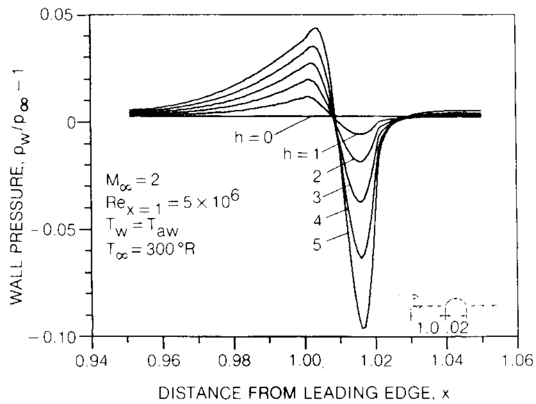

Fig. 6 Bump on flat plate; effect of varied bump height, p_w vs x.

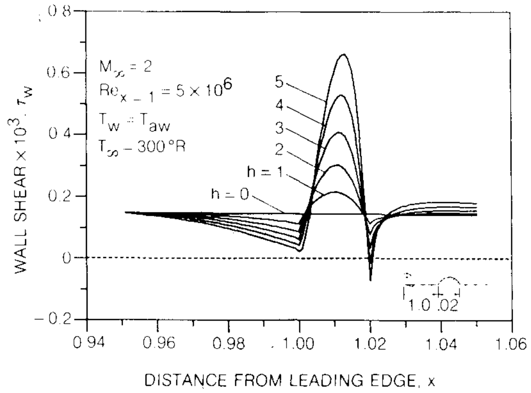

Fig. 7 Bump on flat plate; effect of varied bump height, τ_w vs x.

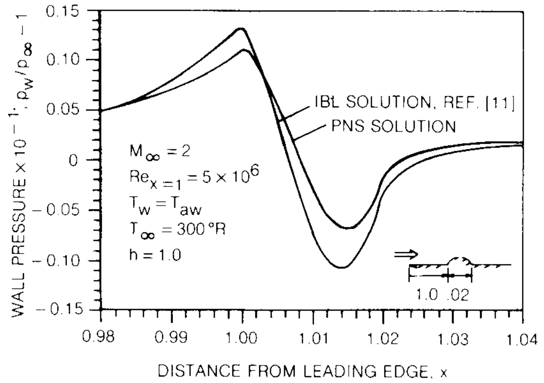

Fig. 8 Bump on flat plate; comparison of PNS and IBL solutions, p_w vs x.

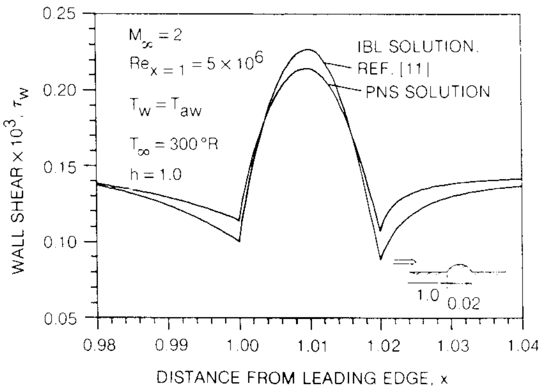

Fig. 9 Bump on flat plate; comparison of PNS and IBL solutions, τ_w vs x.

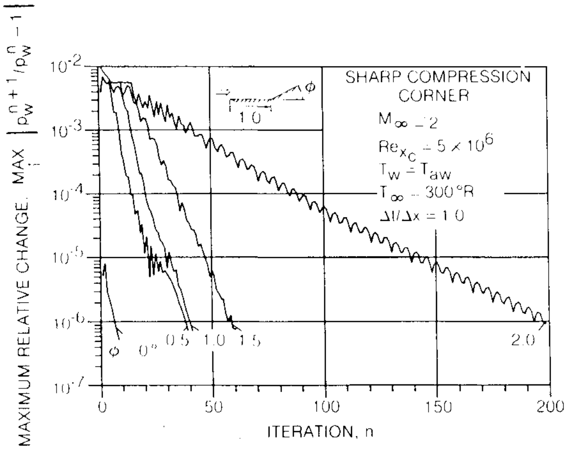

Fig. 10 Convergence histories for sharp compression ramp at various ramp angles. Maximum relative change vs iteration.

Convergence for the unseparated cases of $\phi = 0$, 0.5, and 1.0 deg is seen to occur in less than 40 iterations. The weakly separated case of $\phi = 1.5$ deg converged in less than 60 iterations. For $\phi = 2.0$ deg, where the flow is more extensively separated, convergence is considerably slower. The "scalloped" form which the convergence history assumes is due to the repeated cycling of the maximum error from the rear of the separation bubble to the front of the bubble. All of the cases were run at a constant value of $\Delta t/\Delta x = 1.0$. This is not the optimal value, hence, more rapid convergence is possible.

Concluding Remarks

The ADE procedure presented herein has been applied to several model problems and the results have been discussed. The principal purpose of this study has been to develop and demonstrate the numerical technique. Although it would have been desirable to demonstrate the present procedure on configurations for which experimental data is available, little laminar supersonic data exists. The present method is not limited to laminar flow, however, it was felt that introducing the uncertainties associated with turbulence modeling in order to compare with turbulent experimental data would not have been productive in terms of evaluating the basic numerical procedure. It is also noted that the technique developed here is applicable to hypersonic flow, as has been previously demonstrated for single pass PNS solution techniques (e.g., Ref. 1). The inclusion of real gas effects might have to be considered to properly reflect the physics of some hypersonic flow problems.

The present ADE solution technique allows the parabolized Navier-Stokes equations to be efficiently employed in solving strong interaction problems. Computer storage requirements are minimal because only the pressure needs to be saved in a two-dimensional array, and only in subsonic regions. The need for a higher-order accurate method is indicated by the slow approach of the solution to the limiting solution for $\Delta x \rightarrow 0$. Extension to three-dimensional flows appears to be reasonably straightforward and would yield even more significant savings over full Navier-Stokes solvers than for the two-dimensional flows considered herein.

Acknowledgment

This research was supported by NASA Grant NCA2-OR130-101, NASA Training Grant NGT 36-004-800 (for M. Barnett) and Office of Naval Research Contract N00014-76C-0364.

On January 25, 1986 Professor R.T. Davis passed away. The co-author wishes to express his most heartfelt sorrow at the loss of a friend and mentor. Tom's generosity and candor, his exceptional abilities as a teacher and researcher, and his many important contributions to fluid mechanics will keep his memory fondly alive in his many friends and associates.

References

[1]Rudman, S. and Rubin, S.G., "Hypersonic Viscous Flow Over Slender Bodies with Sharp Leading Edges," *AIAA Journal,* Vol. 6, Oct. 1968, pp. 1883-1889.

[2]Cheng, H.K., Chen, S.Y., Mobley, R., and Huber, C., "On the Hypersonic Leading-Edge Problem in the Merged-Layer Regime," *Advances in Applied Mechanics,* Supplement 5, Vol. 1, 1969, pp. 451-463.

[3]Davis, R.T., "Numerical Solution of the Hypersonic Viscous Shock-Layer Equations," *AIAA Journal,* Vol. 8, May 1970, pp. 843-851.

[4]Vigneron, Y.C., Rakich, J.V., and Tannehill, J.C., "Calculation of Supersonic Viscous Flow Over Delta Wings with Sharp Subsonic Leading Edges," NASA TM-78500, June 1978.

[5]Rubin, S.G., "A Review of Marching Procedures for Parabolized Navier-Stokes Equations," *Numerical and Physical Aspects of Aerodynamic Flows,* Springer-Verlag, New York, 1982, pp. 171-185.

[6]Lin, A. and Rubin, S.G., "Three-Dimensional Supersonic Viscous Flow over a Cone at Incidence," *AIAA Journal,* Vol. 20, Nov. 1982, pp. 1500-1507.

[7]Stewartson, K., "Multistructured Boundary Layers on Flat Plates and Related Bodies," *Advances in Applied Mechanics,* Vol. 14, Academic Press, Inc., 1974, pp. 145-239.

[8]Davis, R.T. and Werle, M.J., "Numerical Methods for Interacting Boundary Layers," *Proceedings of the 1976 Heat Transfer and Fluid Mechanics Institute,* Stanford University Press, Stanford, CA, 1976, pp. 317-339.

[9]Barnett, M., "The Calculation of Supersonic Flows with Strong Viscous-Inviscid Interaction Using the Parabolized Navier-Stokes Equations," Ph.D. Dissertation, Department of Aerospace

Engineering and Applied Mechanics, University of Cincinnati, May 1984.

[10]Barnett, M., Davis R.T., and Rakich, J.V., "Implicit Boundary Conditions for the Solution of the Parabolized Navier-Stokes Equations for Supersonic Flows," *Journal of Computational Physics,* Vol. 48, No. 2, Nov. 1982, pp. 168-181.

[11]Davis, R.T., "A Procedure for Solving the Compressible Interacting Boundary-Layer Equations for Subsonic and Supersonic Flows," AIAA Paper 84-1614, June 1984.

[12]Werle, M.J. and Vatsa, V.N., "A New Method for Supersonic Boundary Layer Separations," *AIAA Journal,* Vol. 12, Nov. 1974, pp. 1491-1497.

[13]Sychev, V.V., "On Laminar Separation," *Mehanika Zhidkosti i Gaza,* No. 3, 1972, pp. 47-59.

[14]Messiter, A.F., "Laminar Separation: A Local Asymptotic Flow Description for Constant Pressure Downstream," *AGARD Conference Proceedings on Flow Separation,* CP-168, Chap. 4, 1975, pp. 1-10.

[15]Davis, R.T., "The Use of Crocco Type Variables in the Solution of Boundary Layer Problems," ARL Rept. TR 74-0059, July 1974.

[16]Carter, J.E., "Numerical Solutions of the Supersonic, Laminar Flow Over a Two-Dimensional Compression Corner," Ph.D. Dissertation, Department of Aerospace Engineering, Virginia Polytechnic Institute and State University, Blacksburg, August 1971.